

# EXAMPLES OF ADMISSIBLE SIMPLIFICATION OF MATHEMATICAL THEORIES

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”Mathematicians, like physicists, are pushed by a strong fascination. Research in mathematics is hard, it is intellectually painful even if it is rewarding, and you wouldnt do it without some strong urge. ” [D. Ruelle]. We shall give some examples from our experience, when we were able to simplify some serious mathematical models to make them understandable by children, preserving both aesthetic and intellectual value. The latter is in particularly measured by whether a given simplification allows setting a sufficient list of problems feasible for school students.

For a more evident demonstration of our method we chose primary school students (6-9 year) as a target group. We give examples of Turing machine, Cellular automata, Minesweeper, Graph theory.

Nowadays, it is thought of importance to introduce current scientific achievements to children by telling them of the black holes or demonstrating some impressive chemical experiments. Can this approach however satisfy us? A. Zvonkin in his acclaimed work [J.Math. Behaviour, 1992] on early child development translates Poincares theory on the role of subconscious into a practical recommendation: *questions are more important than answers*. He depicted an experiment that targeted to find out whether it is using attractive materials that engaged children in his lessons or the lessons per se. Then I say, All right, I have to finish the lesson, but you may play with mosaic. My words are met with an unanimous yell of indignation, No-o, we want a problem!. Thats how I understood what the truth was. *Children*

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*need intellectual/aesthetical pleasure of full value.* If one of the halves is absent, the full value is lost, together with the festive feeling. Thus, if you agree with Poincares theory on the role of subconscious work [Science and Method, 1908], you would agree that preparing a model accompanied by a questionnaire for children will further help them to do research on a higher qualitative level.

The main motivation of the present article is to promote a thesis that the mathematical scientific community needs for a significant volume of the internally generated educational product for pre-college students. Really, studies that are set up for pupils by an active researcher, as a rule, contribute to development of skills essential for successful research activities. Unfortunately, this aspect is excluded from school curriculum scope. Systematical work in this direction is only possible in case when an actively working mathematician has a part of his working time (say 30 percents) solely dedicated to this research.

Many of us introduce our own children to this research, and usually we chose a transphenomenal Socrates' dialog's style. It is of obvious importance both to identify the kernel of these studies discussed and to diversify them so they could be more easily employed by the colleagues when working with students. Here we present our results describing an attempt to create a way to introduce our 7-9 years old pupils to several nontrivial mathematical models. Briefly, we have started with the models that originally could not cause a sustained interest of the child. After a period of time and series of experiments with pupils we were able to fix a simplified model equipped with a list of questions and exercises.

The most valuable exercises are those to allow nontrivial routine multiple repetitions. It is very important for children to gain positive experience of routine exercises. It would allow them to believe/realize that multiple routine exercises in school is a necessary and natural step to the meaningful beauty as in case of a painter's pupil who is ready to grind colours for hours and hours because he is fascinated by the art. Since generation of the quality routine exercises is extremely important per se, we formulate a non-formal law:

*A problem is solvable by a child if and only if his interest exceeds the amount of effort.*

The exercise can be considered successful even in case when pupils are not able to find a solution themselves. *Children develop when confronted with material that has already been able to understand, but not yet to reproduce.* Some of my pupils were able to solve all exercises concerning Cellular automata (Model 1 below), however just an execution of programs for Turing machine (Model 2 below) is done on a peak of their efforts.

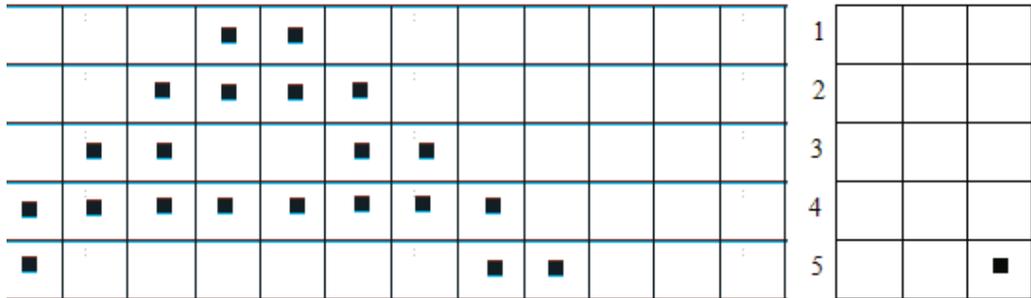


Figure 1: (a). Example of Dynamics.

## 1 Model 1. GAME OF LIFE

Game of Life by John Conway is a well known dynamics with discrete time on the two-dimensional lattice. Of course, the level of abstractness is quite high even to ask children just to find a descendant. We decided to propose them an exercise with 1D cellular automata, which inherits the idea of the "Game of Life". There is a nearest neighbors interaction and life appears/maintained if and only if it has only one neighbor. The example of the dynamics is presented on the fig. 1. The great educational advantage with 2D case was that children could write descendants one below another.

- Continue a given (local) configuration until the 10th descendant.

This exercise is already quite complicated for children, and here we have an effect that Dynamics or Games have magnetic influent to research it. When the pupils reached steady success in solving the exercise one we proposed them to

- try to find the ancestor of given local configuration.

They did it in the following way: first, every pupil painted his own local configuration, then he wrote down the descendant on a piece of paper. This piece of paper was transferred to the pupil's neighbor who tried to reconstruct the initial configuration.

- Continue a given nonlocal periodical configuration until the 10th descendant.
- Find all ancestors of the empty configuration
- Find the stationary point of the dynamics

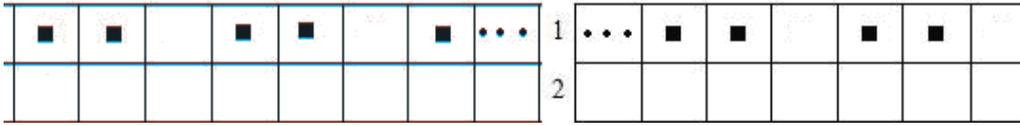


Figure 2: Stationary configuration. Dots mean periodicity.

Note that in this case children perform these exercises with quite abstract objects like *inverse map*, *infinite configuration*, *stationary point*. We believe that exercises with infinite periodical configurations are quite valuable. Here children have to deal with rather abstract objects. In order to figure out a descendant they should imagine the cells that are located beyond their paper sheet. At the same time, such exercises are both constructive and doable

## 2 Model 2. "TURING MACHINE"

For a long time we have been trying to find an admissible realization of the Turing machine. Once we have read a juvenile book "EIGHT CHILDREN AND A TRUCK" by Anne-Cath. Vestly. That is a story about a low-income family consisted of father, mother, 8 children and ... a truck. Truck was a quasi alive object in this story. All of them lived in concord, and Father and the truck had to work a lot, as they worked in logistics. I have decided that this Vestly's world is a good basis for realization of the Turing machine. "...Previously, they lived in the city, in a small apartment, but recently they bought a cabin in the woods, and could even afford their own cow. A father has a lot of worries at home, the truck seemed to become an adult so it could well work a little bit on its own, to free up Father for his family. Father has realized it, they are very well aware of each other, and he began to think how you could he use it, as the truck was still not a human, so he could not just say: "transfer the goods there, then go back to the warehouse ". He could remember only one instruction. Father was a very serious person, who brings everything to the end, no matter what it takes, we know this is not a joke to bring up once the whole eight children Thus, he decided to enumerate all 8 stations that he and the truck were serving" (Fig. 2.(a)). This is an example of the set of instructions for the truck that it needs to do at a given station (Fig 2.(b)). For example, in the first line it reads: if you came to the station, moving to the right (arrow), and you do not have a box in the trunk (the icon with the body blank circle), and there is nothing for you at the station, then you have to (look at the icons after the vertical line) to go blank to the

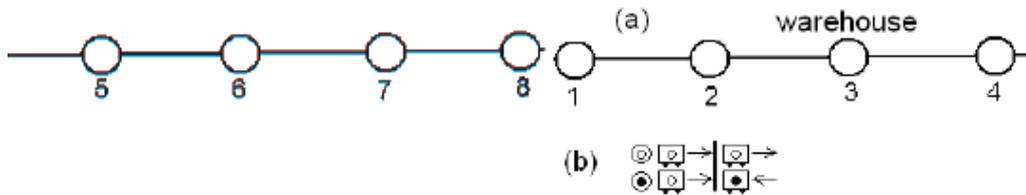


Figure 3: (a) Tape. (b) Example of set of instructions.

right. A second line reads that if there is a box for you at the station, then you should pick it up and go to the previous station. This is an example of the scheme the Father made for each station.

- The box is on the warehouse. Transfer it to the station 4.
- The same, but finally truck should return to the warehouse.
- *Adding.* There are 4 boxes in the warehouse and there are 5 boxes in the 5th station. Truck should transfer all boxes to the warehouse. Father got the same task on the next day but with another amount of boxes. Should he change the program for the truck?
- There is one box at the 2nd station and at the 3d stations. Transfer one box to the 4th station and one box to the 5th station.
- *Bisection.* There are 4 boxes at the warehouse. Transfer a half to the 4th station. How these instructions would work if the amount of boxes at the warehouse was 6? What would be the result in case of an odd number of boxes?
- *Subtraction.* Farther was informed that during the next week, on a daily basis, he should take away from the warehouse that many boxes so that their number becomes equal to the number of boxes at the fourth station.
- *Comparison.* There are boxes at the warehouse and at the 4th station. Determine which station does contain more boxes.
- *Interconversion to the unary system.* There are 5 boxes at the warehouse. Transfer them one by one to the next 5 stations.
- *Invert.* Once it turned out that the boxes were loaded at the warehouse in the wrong order. So farther asked to pull all of them out and reload them back in the reverse order, so that the first box that was previously put in the warehouse should now go last.

Figure 4: Example of the programm written by a 9 years old girl.

### 3 Model 3. MINESWEEPER

The following example is not a *simplification of the model* in its original form. "Minesweeper" is a computer game, quite popular among people that may be very far from science. This was a subject of a number of mathematical researches and not only in terms of the complexity theory: recently it was proved [R. Kaye] that *Turing machine computations can be reduced to infinite minesweeper problems*.

Minesweeper's player always solves nontrivial but not too sophisticated tasks. Evidently this game can serve as a training tool. The problem is unique, as even starting with the same table, players can choose different cells to start with. Thus the teacher cannot control the difficulty level. Here it was decided to develop tables where several cell are open and these open cells determine the solution uniquely. First tables were produced manually, however such method proved to be quite time demanding... Therefore Evgeny Lakshtanov have started developing an automatic production, and several months later he has realized that it takes to analyze spectral properties of discrete Laplacians in order to prepare a table for the primary school children to solve.

An algorithm of table generation: 1. Draw a table  $m \times n$  similar to a chess board. For every "black" cell Bernoulli test is performed and if the result is 1, we put a "mine" in this cell. After having run through all the black cells, we fill all "white" cells with numbers (0-4) that are equal to the

Figure 5: Example of the table for "paper minesweeper".

amount of mines among neighbors.

As a result, we have a table for the "paper minesweeper" and the distribution of mines constructed in the first step of the algorithm being the solution. How to prove that this solution is unique?

**Theorem.**[2, Th. 1]. If numbers  $m + 1$  and  $n + 1$  are coprime, then the constructed table admits one solution only.

## 4 Own mathematics

Before these set of exercises we usually ask pupils if they have own experience to introduce objects in mathematics. We talk to them that everyday they study objects like "numbers, functions, triangles" which were introduced much time ago: (Every pupil need a map of the Europe for next exercises.)

**Definition 1.** Country is *monogamous* if it has exactly one neighbor.

- Find all monogamous country in Europe.

**Definition 2.** A monogamous country is called *happy monogamous* if its neighbor is also monogamous.

- Find all happy monogamous countries in Europe. Why France (Denmark) is not a happy monogamous?

**Definition 3.** A point is called *attractive* if shared by 3 countries.

- Show several attractive points. (Why this point is not attractive?, Has Denmark any attractive points? Which other countries do not have attractive points?)

**Theorem 1.** Monogamous countries do not have attractive points.

- Prove theorem 1.
- Which attractive point is the closest to Portugal?

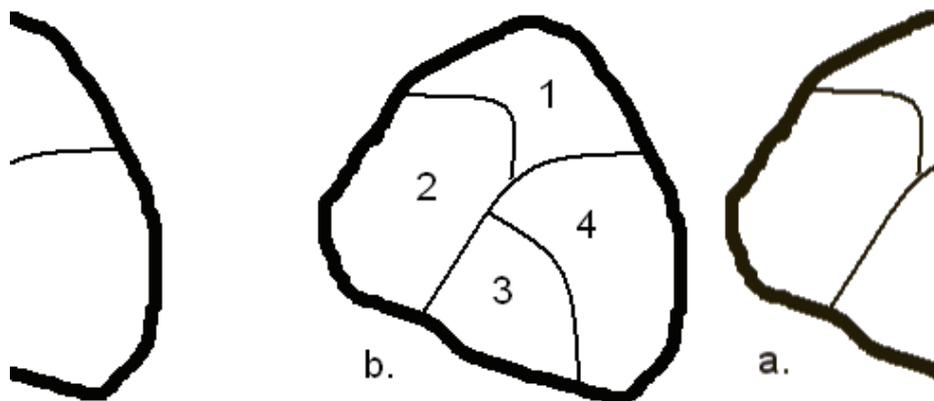


Figure 6: a. There are 3 friendly countries on the island.

- What is the second closest attractive point to Portugal?

Comment. It is better to use a map where Andorra is colored not too distinctively, or at least it should be small enough not to give out the answer too soon that the nearest attractive point to Portugal also belongs to Andorra.

The next set was particularly chosen to give exercises on the *fit four colours* theorem. Children will need a piece of paper and a pen/pencil.

**Definition 4.** A country is called *friendly* if it has at least two neighbors, and each two of them also neighbor each other. Rank of a friendly country is equivalent to a number of its neighbors.

- Here is an island in the sea. Check that all 3 countries on the picture 4.a are *friendly*.

- What countries on the picture 4.b are friendly?
- Paint 4 countries on the island such a way that all of them should be friendly.

Let us return to the map of Europe.

**Definition 5.** The rank of a friendly country equals the number of its neighbours.

- Can rank of a friendly country be 1? What is the rank of Moldova?
- Find all rank 2 friendly countries in Europe. Find all rank 3 friendly countries in the Europe.
- + Can rank of a friendly country be 4?

Some monkeys use tail as a limb. Now, formally we can think that they have an odd number of limbs.

Definition 6. The country is tailed if it has an odd number of neighbours. A country is *tailless* in case this number is even.

•+ Prove that for any map, amount of tailed countries is even.

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